THE GURUKUL INSTITUTE

PLOT 5C, 2ND FLOOR, GANAPATI COMPLEX, SEC-13, OPP. JAIPURIA SCHOOL, VASUNDHARA, GHAZIABAD (U.P)

Q1 If $P(x) = ax^2 + bx + c$, and $Q(x) = -ax^2 + bx + c$, where ac not equal to zero. Show that the equation $P(x).Q(x) = 0$ has at least two real roots. (IIT 89)

Solution: Roots of equation $P(x).Q(x) = 0$ will be the roots of equation $P(x)=0$ and $Q(x)=0$

D_1 be the discriminant of $P(x)$
D_2 be the discriminant of $Q(x)$

$D_1 + D_2 = (b^2 - 4ac) + (b^2 + 4ac) = 2b^2 \geq 0$

So at least one of D_1 and D_2 must be greater than or equal to zero.

So at least one of the equations $P(x)=0$, and $Q(x)=0$ has real roots.

Hence the equation $P(x).Q(x)=0$ has at least two real roots.

second method:

ac is not equal to zero.
So $ac>0$, or $ac<0$
If $ac>0$, $D_2 = b^2 + 4ac > 0$
If $ac<0$, $D_1 = b^2 - 4ac > 0$

So at least one of D_1 and D_2 must be greater than or equal to zero.

So at least one of the equations $P(x)=0$, and $Q(x)=0$ has real roots.

Hence the equation $P(x).Q(x)=0$ has at least two real roots.

Q.2. Show that if p, q,r,s are real numbers and $pr=2(q+s)$ then at least one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.

Solution:

$x^2 + px + q = 0$ ....(1)

$x^2 + rx + s = 0$ ....(2)

$pr=2(q+s)$ ....(3)

Let D_1 and D_2 be the discriminants of equations (1) and (2) respectively.

$D_1 + D_2 = p^2 - 4q + r^2 - 4s$

$= p^2 + r^2 - 4(q+s)$

$= p^2 + r^2 - 2pr$ (using equation (3))

$= (p-r)^2 >= 0$

So at least one of D_1 and D_2 $>=0$

So at least one of the equations (1) and (2) have real roots.

3. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. (IIT 1982)

Let $y= e^{\sin x}$.
Then the equation becomes $y - 1/y - 4 = 0$

or, $y^2 - 4y - 1 = 0$

Discriminant, $D = 4^2 - 4(1)(-1) = 20$

Two roots are $y = (4 + \sqrt{20})/2$, $(4 - \sqrt{20})/2$

or, $y= (2+\sqrt{5})$, $(2-\sqrt{5})$

or, $e^{\sin x} = (2+\sqrt{5})$, $(2-\sqrt{5})$

Taking log on both sides, we get

$\sin x = \log(2+\sqrt{5})$, or $\sin x=\log(2-\sqrt{5})$

$2-\sqrt{5}$ is negative. But log of negative number is not defined.

So only one equation is left.
$\sin x= \log(2+\sqrt{5}) > \log e$
or, $\sin x>1$
This is not possible.
So the required equation has no real solution.

If α and β are the roots of the equation $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$, where n is an even integer, prove that α/β , β/α are the roots of the equation $x^n + 1 + (x+1)^n = 0$.

Solution:

α and β are the roots of the equation $x^2 + px + q = 0$
So, $\alpha + \beta = -p$.........(1)
$\alpha \beta = q$ ....(2)
Since α and β are the roots of the equation $x^{2n} + p^n x^n + q^n = 0$
$\alpha^{2n} + p^n \alpha^n + q^n = 0$
or, $(\alpha^n)^2 + p^n \alpha^n + q^n = 0$..........(3)

and,
$(\beta^n)^2 + p^n \beta^n + q^n = 0$..............(4)
From (3) and (4) we see that are the roots of $y^2 + p^n y + q^n = 0$
So, $\alpha^n + \beta^n = -p^n$.........(5)
$\alpha^n \beta^n = q^n$ ....(6)

From (1), we have
$\alpha + \beta = -p$
or, $(\alpha + \beta)^n = (-p)^n = p^n$ (n is even)

or, $(\alpha + \beta)^n = -(-p^n) = -(\alpha^n + \beta^n)$ (from 5)

or, $\alpha^n + \beta^n + (\alpha + \beta)^n = 0$ ....(7)

Dividing (7) by α^n, we get

$(\alpha/\beta)^n + 1 + ((\alpha/\beta) + 1)^n = 0$ ....(8)

Dividing (7) by β^n, we get

$(\beta/\alpha)^n + 1 + ((\beta/\alpha) + 1)^n = 0$ ....(9)

From (8) and (9) we see that α/β and β/α are the roots of
$x^n + 1 + (x+1)^n = 0$.

2. If sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, show that bc^2, ca^2, ab^2 are in AP. (IIT 76)

Solution:

Let α and β be the roots of the equation $ax^2 + bx + c = 0$.
Then, $\alpha + \beta = -b/a$ ....(1)

$\alpha \beta = c/a$ ....(2)

Now, $\alpha + \beta = 1/\alpha^2 + 1/\beta^2$

or, $\alpha + \beta = (\alpha^2 + \beta^2)/(\alpha \beta)^2$

or, $\alpha + \beta = ((\alpha + \beta)^2 - 2\alpha\beta) / (\alpha \beta)^2$ ....(3)

Now put the values of $(\alpha + \beta)$ and $(\alpha \beta)$ from (1) and (2) in equation (3).

Thus, we get

$-b/a = (b^2 - 2ac)/ c^2$

or, $-bc^2 = b^2a - 2ca^2$

or, $bc^2 + ab^2 = 2ca^2$

Hence bc^2 , ca^2 , ab^2 are in A.P.
Proved.

Q4 . If one root of a quadratic equation $ax^2 + bx + c = 0$ is equal to nth power of the other, show that

$(ac^n)^{1/n+1} + (a^n c)^{1/n+1} + b = 0$.

Q5 If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that

$(r+1)^2 / r = b^2 / ac$

Q6 Solve for x:

$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$

Q7 If c,d are the roots of the equation$(x- a)(x-b) - k = 0$, show that
a, b are the roots of the equation $(x-c)(x-d) + k = 0$.

Q8 The coefficient of x in the equation $x^2 + px + q = 0$ was wrongly written as 17 in place of 13 and the roots thus found were -2 and -15. Find the roots of the correct equation.